Further results on the Craig-Sakamoto Equation

John Maroulas[1]

February 2, 2008

Abstract

In this paper necessary and sufficient conditions are stated for the Craig-Sakamoto equation $det(I - sA - tB) = det(I - sA)\, det(I - tB)$, for all scalars s, t. Moreover, spectral properties for A and B are investigated.

1 Introduction

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with elements in $\mathbb{C}$. For A and $B \in M_n(\mathbb{C})$, the well known in Statistícts [1] Craig-Sakamoto (CS) equation

$$det(I - sA - tB) = det(I - sA)\, det(I - tB) \tag{1}$$

for all scalars s, t has occupied several researchers. In particular, in [5] O. Trussky presented that the CS equation is equivalent to $AB = O$, when A, B are normal and most recently in [4] Olkin and in [2] Li proved the same result in a different way. The author, together with M. Tsatsomero and P. Psarrako in [3], have investigated the CS equation involving the eigenspaces of A, B and $sA + tB$. Being more specific, if $\sigma(X)$ denotes the spectrum for a matrix X, $m_X(\lambda)$ the algebraic multiplicity of $\lambda \in \sigma(X)$, and $E_X(\lambda)$ the generalized eigenspace corresponding to λ, we have shown in [3]:

[1]Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, GREECE. E-mail:maroulas@math.ntua.gr. This work is supported by a grant of the EPEAEK, project "Pythagoras II".

Proposition 1 *For the $n \times n$ matrices A, B the following are equivalent :*

I. *The CS equation holds*

II. *for every $s, t \in \mathbb{C}$, $\sigma(sA \oplus tB) = \sigma\left((sa + tB) \oplus O_n\right)$, where O_n denotes the zero matrix*

III. $\sigma(sA + tB) = \{ s\mu_i + t\nu_i : \mu_i \in \sigma(A), \ \ \nu_i \in \sigma(B) \}$, *where the pairing of eigenvalues requires either $\mu_i = 0$ or $\nu_i = 0$.*

Proposition 2 *Let the $n \times n$ matrices A, B satisfy the CS equation. Then,*

I. $m_A(0) + m_B(0) \geq n$.

II. *If A is nonsingular, then B must be nilpotent.*

III. *If $\lambda = 0$ is semisimple eigenvalue of A and B, then $rank(A) + rank(B) \leq n$.*

Proposition 3 *Let $\lambda = 0$ be semisimple eigenvalue of $n \times n$ matrices A and B such that $BE_A(0) \subset E_A(0)$. Then the following are equivalent.*

I. *Condition CS holds.*

II. $\mathbb{C}^n = E_A(0) + E_B(0)$.

III. $AB = O$.

The remaining results in [3] are based on the basic assumption that $\lambda = 0$ is a semisimple eigenvalue of A and B. Relaxing this restriction, we shall attempt here to look at the CS equation focused on the factorization of polynomial of two variables $f(s, t) = det(I - sA - tB)$. Also, considering the determinants in (1), new conditions necessary and sufficient on CS property are stated.

2 Spectral results

The first statement on the CS property is obtained investigating the determinantal equation through the Theory of Polynomials. By Proposition 2 **II**, it is clear that the CS equation is worth valuable when the $n \times n$ matrices A and B are singular. Especially, we define that
" *A and B are called r-***complementary***, if and only if* at most, *r rows (columns), $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ of A are shifted and substituted by the corresponding $b_{i_1}, b_{i_2}, \cdots, b_{i_r}$ rows (columns) of B, such that the structured matrix $N(i_1, i_2, \cdots, i_r)$ of a' s and b' s rows is nonsingular.*"

Note that, $n - r \leq rank(B)$.

For example, the pair of matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not 1 or $2-$complementary, on behalf of $rank \begin{bmatrix} A \\ B \end{bmatrix} = 3,$ but the pair

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{B} = B$$

is 1-complementary and not 2-complementary, since $detN(b_1, a_2, a_3) = det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$ and $detN(b_1, b_2, a_3) = det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 0.$

Proposition 4 *Let the* $n \times n$ *singular matrices* A *and* B *be* $[n - m_B(0)]$*-complementary with* $\theta = \sum_{i_1, \ldots, i_{n-m_B(0)}} detN(i_1, i_2, \ldots, i_{n-m_B(0)}) \neq 0,$ *where the sum is over all possible combinations* $i_1, \ldots, i_{n-m_B(0)}$ *of* $n - m_B(0)$ *of the indices* $1, 2, \ldots, n.$ *If they satisfy the CS equation, then*

$$m_A(0) + m_B(0) \; = \; n.$$

Proof. Let $rankB = b\,(< n).$ Then $\lambda = 0$ is eigenvalue of B with algebraic multiplicity $m_B(0) = m \geq n - b.$ Denoting

$$\beta(t) \doteqdot det(tI - B) = t^n + \beta_1 t^{n-1} + \cdots + \beta_{n-m} t^m,$$

where $\beta_k = (-1)^k \sum B_k$ and B_k are the $k \times k$ principal minors of $B,$ then

$$\begin{aligned} det(tB - I) \;&=\; (-1)^n \, t^n \, det(t^{-1} I - B) \\ &=\; (-1)^n \, \left(1 + \beta_1 \, t + \cdots + \beta_{n-m} \, t^{n-m}\right). \end{aligned}$$

The polynomial $\widetilde{\beta}(t) \;=\; 1 + \beta_1 \, t + \; \cdots + \beta_{n-m} \, t^{n-m}$ has precisely $n - m$ nonzero roots, let $t_1, t_2, \cdots, t_{n-m},$ since $\widetilde{\beta}(0) = 1 \neq 0.$ Moreover, we have

$$det(sA + tB - I) = |A| s^n + f_1(t) s^{n-1} + \cdots + f_{n-1}(t) s + |tB - I|, \tag{2}$$

where

$$f_1(t)=\sum_i det\hat{A}_i, \quad \text{with} \quad \hat{A}_i=\begin{bmatrix} a_{11} & & \cdots & & a_{1n} \\ \vdots & & & & \vdots \\ tb_{i1} & \cdots & tb_{ii}-1 & \cdots & tb_{in} \\ \vdots & & & & \vdots \\ a_{n1} & & \cdots & & a_{nn} \end{bmatrix}.$$

Note that, $\widehat{A}_i$ arises by A when the $i-$row of A is substituted by the $i-$row of $tB-I$. Similarly,

$$f_2(t)=\sum_{i,j} det\hat{A}_{ij}, \quad \text{with} \quad \hat{A}_{ij}=\begin{bmatrix} a_{11} & & \cdots & & & a_{1n} \\ \vdots & & & & & \vdots \\ tb_{i1} & \cdots & tb_{ii}-1 & \cdots & & tb_{in} \\ \vdots & & \ddots & & & \vdots \\ tb_{j1} & \cdots & & tb_{jj}-1 & \cdots & tb_{jn} \\ \vdots & & & & & \vdots \\ a_{n1} & & \cdots & & & a_{nn} \end{bmatrix},$$

and $\hat{A}_{ij}$ is obtained by A, substituting the i and j rows of A by the corresponding rows of $tB-I$. The summation in $f_2(t)$ is referred to all pairs of indices i,j by $\{1,2,\ldots,n\}$. Hence, by the equation (2) and the CS equation

$$(-1)^n\, det(sA+tB-I)=det(sA-I)\, det(tB-I), \qquad \forall\ s,\, t$$

for $t=t_1,\, t_2,\, \cdots, t_{n-m}$, we obtain

$$|A|s^n+f_1(t_i)s^{n-1}+\cdots+f_{n-1}(t_i)s=0, \qquad \forall \ \ s$$

and consequently

$$|A|=0 \quad , \quad f_1(t_i)=f_2(t_i)=\cdots=f_{n-1}(t_i)=0, \quad \text{for} \quad i=1,2,\ldots,n-m\,. \tag{3}$$

Due to the matrices A and B are $[n-m_B(0)]$-complementary and the leading coefficient of $f_{n-m}(t)$ is equal to the nonzero θ, then $deg(f_{n-m}(t))=n-m$ and $deg(f_k(t))\leq n-m$, for $k=1,2,\ldots,n-m-1$. Moreover, by (3) we have

$$f_1(t)=f_2(t)=\cdots=f_{n-m-1}(t)=0, \quad \forall \ \ t$$

Reminding that A_ℓ denotes the $\ell\times\ell$ principal minor of A, by $f_1(t)=0$, clearly

$$f_1(0)=\sum A_{n-1}=0 \implies c_{n-1}=0.$$

Similarly, by

$$
\begin{aligned}
& f_2(t)=0 \Longrightarrow \sum A_{n-2}=0 \quad \Longrightarrow \quad c_{n-2}=0 \\
& \quad \vdots \\
& f_{n-m-1}(t)=0 \quad \Longrightarrow \quad \sum A_{m+1}=0 \quad \Longrightarrow \quad c_{m+1}=0,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\delta_A(\lambda)=|\lambda I-A| & = \lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+(-1)^n|A| \\
& = \lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+(-1)^m c_m\lambda^{n-m}. \qquad (4)
\end{aligned}
$$

In (4), $c_m \neq 0$, since $(-1)^{n-m}c_m = \theta t_1 t_2 \cdots t_{n-m}$. Thus, $\lambda=0$ is eigenvalue of A with algebraic multiplicity $n-m_B(0)$, whereby we conclude

$$
m_A(0)+m_B(0) \;=\; n.
$$

□

Remark 1 By the proof of Proposition 4, it is evident that the equality $m_A(0)+m_B(0) \;=\; n$ holds, when the matrices B and A are $[n-m_A(0)]$-complementary and

$$
\theta=\sum_{j_1,\ldots,j_{n-m_A(0)}} detN(j_1,j_2,\ldots,j_{n-m_A(0)}) \neq 0.
$$

Corollary 1 *Let the $n\times n$ singular and $[n-m_B(0)]$-complementary matrices A and B. If $\theta \neq 0$ and these matrices satisfy the CS equation (1), then*

I. $\lambda=0$ *is semisimple eigenvalue of A and B $\Longrightarrow rankA+rankB=n$.*

II. $\lambda=0$ *is semisimple eigenvalue of A $\Longrightarrow rankA=m_B(0)$.*

Proof. I. Because

$$
n-rankA \leq m_A(0)=n-m_B(0),
$$

we have $rankA+rankB \geq m_B(0)+r \geq n$. Hence, by **III**, Proposition 2, we obtain the equality.
II. By the assumption and Proposition 4 we have $rankA=n-m_A(0)=m_B(0)$. □

Closing this section, we present a property of generalized eigenspaces of nonzero eigenvalues of A and B.

Proposition 5 *Let* $\lambda = 0$ *be semisimple eigenvalue of* $n \times n$ *matrices* A *and* B *such that* $E_A(0) + E_B(0) = \mathbb{C}^n$. *If for any* $\lambda \in \sigma(A)\backslash\{0\}$ *, (or,* $\mu \in \sigma(B)\backslash\{0\}$ *), the corresponding generalized eigenspaces* $E_A(\lambda)$*, (*$E_B(\mu)$ *) satisfy* $E_A(\lambda) \subseteq E_B(0)$*, (or,* $E_B(\mu) \subseteq E_A(0)$ *), then*

I. A, B *have the CS property.*

II. $E_A(\lambda) = E_{I-sA-tB}(1 - s\lambda)$*, and* $E_B(\mu) = E_{I-sA-tB}(1 - t\mu)$.

Proof. **I.** Since $E_A(\lambda) \subseteq E_B(0)$, for every $w = w_1 + w_2 \in \mathbb{C}^n$, where $w_1 \in \bigoplus_\lambda E_A(\lambda)$, $w_2 \in E_A(0)$, we have $BAw = BA(w_1 + w_2) = BAw_1 = 0$. Thus, $BA = O$ and consequently $AE_B(0) \subseteq E_A(0)$. The assumption $E_A(0) + E_B(0) = \mathbb{C}^n$, and Proposition 3, lead to the statement **I**.

II. Let $\lambda \in \sigma(A)\backslash\{0\}$, and $x_k \in E_A(\lambda)$ be generalized eigenvector of A of order k. By the assumption, $x_k \in E_B(0)$, and yields

$$\begin{aligned}(I - sA - tB)x_k &= (I - sA)x_k = x_k - s(\,\lambda x_k + x_{k-1}\,)\\ &= (1 - s\lambda)x_k - sx_{k-1}.\end{aligned}$$

Thus, for all chain $x_1, \ldots, x_k, \ldots, x_\tau$ of λ, we have

$$(I - sA - tB)\begin{bmatrix} x_1 & \ldots & x_\tau \end{bmatrix} = \begin{bmatrix} x_1 & \ldots & x_\tau \end{bmatrix} \begin{bmatrix} 1-s\lambda & -s & & & \\ 0 & 1-s\lambda & -s & O & \\ \vdots & & \ddots & \ddots & \\ & & & 1-s\lambda & -s \\ 0 & 0 & & & 1-s\lambda \end{bmatrix}_{\tau\times\tau} \tag{5}$$

Moreover, by the statement **III** in Proposition 1, $s\lambda$ and $t\mu \in \sigma(sA + tB)$. The equivalence of CS equation and $\mathbb{C}^n = E_A(0) + E_B(0)$ in Proposition 3 and the assumption $E_A(\lambda) \subseteq E_B(0)$, lead to $E_B(\mu) \subseteq E_A(0)$. Similarly, if $y_\ell \in E_B(\mu)$ is generalized eigenvector of order ℓ, then $y_\ell \in E_A(0)$ and

$$\begin{aligned}(I - sA - tB)y_\ell &= (I - tB)y_\ell = y_\ell - t(\,\mu y_\ell + y_{\ell-1}\,)\\ &= (1 - t\mu)y_\ell - ty_{\ell-1},\end{aligned}$$

and for all chain $y_1, \ldots, y_\ell, \ldots, y_\sigma$ we obtain

$$(I - sA - tB)\begin{bmatrix} y_1 & \ldots & y_\sigma \end{bmatrix} = \begin{bmatrix} y_1 & \ldots & y_\sigma \end{bmatrix} \begin{bmatrix} 1-t\mu & -t & & & \\ 0 & 1-t\mu & -t & O & \\ \vdots & & \ddots & \ddots & \\ & & & 1-t\mu & -t \\ 0 & 0 & & & 1-t\mu \end{bmatrix}_{\sigma\times\sigma} \tag{6}$$

Clearly, by (5) and (6) are implied the equations in **II**, for any s, t. $\square$

Remark 2 For $z \in E_A(0) \bigcap E_B(0)$ obviously $(I - sA - tB)z = z, \ \forall s, t$. Therefore by the above proposition the Jordan canonical form of $I - sA - tB$, and the matrix

$$F = I_\nu \bigoplus_{\lambda_A \neq 0} \begin{bmatrix} 1 - s\lambda_A & -s & & O \\ & 1 - s\lambda_A & \ddots & \\ & & \ddots & -s \\ O & & & 1 - s\lambda_A \end{bmatrix} \bigoplus_{\mu_B \neq 0} \begin{bmatrix} 1 - t\mu_B & -t & & O \\ & 1 - t\mu_B & \ddots & \\ & & \ddots & -t \\ O & & & 1 - t\mu_B \end{bmatrix},$$

are similar.

The order ν of submatrix I_ν of F declares the number of linear independent eigenvectors which correspond to the eigenvalue $\lambda = 1$ of $I - sA - tB$. Clearly, theses eigenvectors belong to $E_B(0) \backslash E_A(\lambda)$, $E_A(0) \backslash E_B(\mu)$, and $E_A(0) \bigcap E_B(0)$, and ν is equal to

$$\nu = n - (rankA + rankB) = n - \left(dim \bigcup_{\lambda \neq 0} E_A(\lambda) + dim \bigcup_{\mu \neq 0} E_B(\mu) \right).$$

3 Criteria for CS equation

Let

$$f(s,t) = det(I - sA - tB) = \sum_{p,q=0}^{n} m_{pq} s^p t^q, \qquad p + q \leq n. \tag{7}$$

Denoting by $x = \begin{bmatrix} 1 & s & s^2 & \cdots & s^n \end{bmatrix}^T$, $y = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^T$, then (7) is written obviously

$$f(s,t) = x^T M y,$$

where $M = [m_{pq}]_{p,q=0}^{n}$, with $m_{00} = 1$.

Proposition 6 *Let A, $B \in M_n(\mathbb{C})$. The CS equation holds for the pair of matrices A and B if and only if $rankM = 1$.*

Proof. Let A and B managed by the CS property. Then the equation (1) is formulated as

$$x^T M y = x^T a \, b^T y, \tag{8}$$

where

$$a = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_0 \end{bmatrix}^T, \qquad b = \begin{bmatrix} 1 & b_{n-1} & \cdots & b_0 \end{bmatrix}^T,$$

and a_i, b_i are the coefficients of characterictic polynomials

$$det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0, \quad det(\lambda I - B) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_0.$$

Hence, by (8) for any $s_1 \neq s_2 \neq \cdots \neq s_{n+1}$ and $t_1 \neq t_2 \neq \cdots \neq t_{n+1}$ we have

$$V^T \left(M - a\, b^T\right) W = O, \tag{9}$$

where

$$V = \begin{bmatrix} 1 & \cdots & 1 \\ s_1 & \cdots & s_{n+1} \\ \vdots & & \vdots \\ s_1^n & \cdots & s_{n+1}^n \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_{n+1} \\ \vdots & & \vdots \\ t_1^n & \cdots & t_{n+1}^n \end{bmatrix}.$$

Clearly, by (9), we recognize that $M = a\, b^T$, i.e., $rankM = 1$.

Conversely, if $rankM = 1$, then $M = k\, \ell^T$, where the vectors $k,\, \ell \in \mathbb{C}^{n+1}$. Therefore,

$$f(s,t) = x^T M y = x^T k\, \ell^T y = k(s)\ell(t),$$

where $k(s)$ and $\ell(t)$ are polynomials. Since, $f(0,0) = 1 = k(0)\ell(0)$, and

$$\begin{aligned} det(I - sA) &= f(s,0) = k(s)\ell(0), \\ det(I - tB) &= f(0,t) = k(0)\ell(t) \end{aligned}$$

clearly,

$$f(s,t) = k(s)\ell(0)k(0)\ell(t) = det(I - sA)\, det(I - tB).$$

□

Example 1 Let the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma & 1 \\ 0 & 0 & 1-\gamma \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \gamma & 0 \\ 1/\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} f(s,t) = det(I - sA - tB) &= 1 + 2(\gamma-1)s + (\gamma-1)^2 s^2 - t^2 + (1-\gamma)t^2 s \\ &= x^T \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2(\gamma-1) & 0 & 1-\gamma & 0 \\ \gamma-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y \end{aligned}$$

and

$$det(I - sA) = (1 + (\gamma - 1)s)^2\,, \quad det(I - tB) = 1 - t^2.$$

By the criterion (Proposition 6) easily we recognize that A, B have the CS property only for $\gamma = 1$.

Remark 3 In equation (8), if $b^T a = 0$ then $M^2 = 0$, and $M\left(\frac{1}{\|b\|^2}\, b\right) = a$. Therefore,

$$M = P \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix} P^{-1}$$

where $P = \begin{bmatrix} a & p_2 & \cdots & p_{n-1} & \frac{1}{\|b\|^2} b \end{bmatrix}$ and $p_k, \ldots, p_{n-1}$ is an orthonormal basis of $span\{a, b\}^{\perp}$. Then $P^{-1} = \begin{bmatrix} \frac{1}{\|a\|^2} a & p_2 & \cdots & p_{n-1} & b \end{bmatrix}^T$.

Following we note by $M\begin{pmatrix} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{pmatrix}$ the leading principal minor of order $p + q\,(\leq n)$, which is defined by the $i_1, \ldots, i_p$ rows of A and $j_1, \ldots, j_q$ rows of B, i.e.,

$$M\begin{pmatrix} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{pmatrix} = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & a_{i_1 j_1} & a_{i_1 i_3} & \cdots & a_{i_1 j_q} & \cdots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & a_{i_2 j_1} & a_{i_2 i_3} & \cdots & a_{i_2 j_q} & \cdots & a_{i_2 i_p} \\ b_{j_1 i_1} & b_{j_1 i_2} & b_{j_1 j_1} & b_{j_1 i_3} & \cdots & b_{j_1 j_q} & \cdots & b_{j_1 i_p} \\ a_{i_3 i_1} & a_{i_3 i_2} & a_{i_3 j_1} & a_{i_3 i_3} & & & & \vdots \\ \vdots & \vdots & & & \ddots & & & \\ b_{j_q i_1} & b_{j_q i_2} & & & & b_{j_q j_q} & & \\ \vdots & \vdots & & & & & \ddots & \\ a_{i_p i_1} & a_{i_p i_2} & & \cdots & & & & a_{i_p i_p} \end{vmatrix}$$

for $i_1 < i_2 < j_1 < i_3 < \cdots < j_q < \cdots < i_p$. Thus, we clarify a determinental expression of coefficients m_{pq} in (7):

$$m_{pq} = (-1)^{p+q} \sum_{1 \leq i_1 < j_1 < \cdots < j_q < i_p \leq n} M\begin{pmatrix} a_{i_1, \ldots, i_p} \\ b_{j_1, \ldots, j_q} \end{pmatrix}, \quad m_{00} = 1. \tag{10}$$

For example, for $n \times n$ matrices A and B the coefficients of t, st, s^2 and $s^2 t$ are respectively equal to

$$m_{01} = -\sum_{1 \leq j \leq n} M(b_j) = -\,(b_{11} + b_{22} + \cdots + b_{nn}) = -trB$$

$$m_{11} = \sum_{1\le i<j\le n} M\begin{pmatrix} a_i \\ b_j \end{pmatrix} = \sum_{\substack{i,j=1 \\ i<j}}^{n} \left(\begin{vmatrix} a_{ii} & a_{ij} \\ b_{ji} & b_{jj} \end{vmatrix} + \begin{vmatrix} b_{ii} & b_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \right)$$

$$m_{20} = \sum_{1\le i,j\le n} M(a_{ij}) = \sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

and

$$m_{21} = -\sum_{1\le i\le j\le k\le n} M\begin{pmatrix} a_{i,j} \\ b_k \end{pmatrix} = -\sum_{i\le j\le k\le n} \left(\begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{vmatrix} + \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ b_{ji} & b_{jj} & b_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} + \begin{vmatrix} b_{ii} & b_{ij} & b_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \right).$$

Hence, for the matrix M in (7) we have:
$M =$

$$\begin{bmatrix} 1 & -\sum M(b_j) & \sum M(b_{i,j}) & \cdots & (-1)^{n-1}\sum M(b_{j_1,\ldots,j_{n-1}}) & (-1)^n|B| \\ -\sum M(a_i) & \sum M\begin{pmatrix} a_i \\ b_j \end{pmatrix} & -\sum M\begin{pmatrix} a_i \\ b_{j_1,j_2} \end{pmatrix} & \cdots & (-1)^n \sum M\begin{pmatrix} a_i \\ b_{j_1,\ldots,j_{n-1}} \end{pmatrix} & 0 \\ \sum M(a_{i_1,i_2}) & -\sum M\begin{pmatrix} a_{i_1,i_2} \\ b_{j_1} \end{pmatrix} & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & & & \\ \vdots & (-1)^n \sum M\begin{pmatrix} a_{i_1,\ldots,i_{n-1}} \\ b_j \end{pmatrix} & 0 & \cdots & & 0 \\ (-1)^n|A| & 0 & 0 & \cdots & & 0 \end{bmatrix}$$

The zeros in M correspond to the coefficients of monomials of $f(s,t)$ with degree $\ge n+1$. These terms are not presented in $det(I - sA - tB)$, since by (10) the order of principal minors is greater than n. Moreover, the dimension of M in (7) should be less than $n+1$, since the CS equation make sense for singular matrices.

Using the criterion in Proposition 6 in the above formulation of M, it is clear the next necessary and sufficient conditions.

Proposition 7 *The $n \times n$ matrices A and B have the CS property if and only if*

$$\sum M(a_{i_1,\dots,i_p}) \sum M(b_{j_1,\dots,j_q}) = \sum M \begin{pmatrix} a_{i_1,\dots,i_p} \\ b_{j_1,\dots,j_q} \end{pmatrix}, \qquad \textit{for} \quad p+q \leq n,$$

and (11)

$$\sum M(a_{i_1,\dots,i_p}) \sum M(b_{j_1,\dots,j_q}) = 0, \qquad \textit{for} \quad p+q > n.$$

Example 2 In (1) let A be a nilpotent matrix. Then,

$$\sum M(a_i) = \sum M(a_{i,j}) = \cdots = |A| = 0,$$

and by Proposition 7 clearly

$$\sum M \begin{pmatrix} a_{i_1,\dots,i_p} \\ b_{j_1,\dots,j_q} \end{pmatrix} = 0 \quad ; \quad p, q = 1, 2, \dots, n-1.$$

In this case, $M = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & b_{n-1} & \cdots & b_1 & b_0 \end{bmatrix}.$

The equations (11) give also an answer to the problem "*For the $n \times n$ matrix A, clarify the set $CS(A) = \{ B : \quad A$ and B follow the CS property$\}$.*

If $a(s) = det(I - sA)$ and $b(t) = det(I - tB)$, easily we turn out the μ-th order derivative of polynomials at the origin

$$\frac{1}{p!}\, a^{(p)}(0) = \sum M(a_{i_1,\dots,i_p}), \qquad \frac{1}{q!}\, b^{(t)}(0) = \sum M(b_{j_1,\dots,j_q}),$$

and even

$$\frac{1}{p!q!}\, \frac{\partial^{p+q} f(0,0)}{\partial s^p\, \partial t^q} = \sum M \begin{pmatrix} a_{i_1,\dots,i_p} \\ b_{j_1,\dots,j_q} \end{pmatrix}.$$

Thus, if we use the Taylor's expansion of polynomials in (1), by the relationships

$$a^{(p)}(0)\, b^{(q)}(0) = \frac{\partial^{p+q} f(0,0)}{\partial s^p\, \partial t^q}, \qquad \text{for} \quad p+q \leq n,$$

$$a^{(p)}(0)\, b^{(q)}(0) = 0, \quad \text{for} \quad p+q > n,$$

the equations (11) arise again.

References

[1] M. Dumais ans G.P. Styan, A bibliography on the distribution of quadratic forms in normal variables, with special emphasis on the Craig-Sakamoto theorem and on Cochran's theorem, In George Styan ed., *Three Bibliographies and a Guide*, Seventh International Workshop on Matrices and Statistics, Fort Lauderdale, 1-9, 1988.

[2] C-K. Li, A simple proof of the Craig-Sakamoto Theorem, *Linear Algebra and Its Applications*, **321**, (2000), 281-283.

[3] J. Maroulas, P. Psarrakos and M. Tsatsomeros, Separable characteristic polynomials of pencils and property L, *Electronic Journal of Linear Algebra*, **7**, (2000), 182-190.

[4] I. Olkin, A determinantal proof of the Craig-Sakamoto Theorem, *Linear Algebra and Its Applications*, **264**, (1997), 217-223.

[5] O. Trussky, On a matrix theorem of A.T. Craig and H. Hotelling, *Indigationes Mathematicae*, **20**, (1958), 139-141.